THOMAS' CALCULUS

Twelfth Edition

Based on the original work by
George B. Thomas, Jr.
Massachusetts Institute of Technology

as revised by

Maurice D. Weir
Naval Postgraduate School

Joel Hass
University of California, Davis

Addison-Wesley
Boston Columbus Indianapolis New York San Francisco Upper Saddle River
Amsterdam Cape Town Dubai London Madrid Milan Munich Paris Montréal Toronto
Delhi Mexico City São Paulo Sydney Hong Kong Seoul Singapore Taipei Tokyo

Editor-in-Chief: Deirdre Lynch
Senior Acquisitions Editor: William Hoffman
Senior Project Editor: Rachel S. Reeve
Associate Editor: Caroline Celano
Associate Project Editor: Leah Goldberg
Senior Managing Editor: Karen Wernholm
Senior Production Supervisor: Sheila Spinney
Senior Design Supervisor: Andrea Nix
Digital Assets Manager: Marianne Groth
Media Producer: Lin Mahoney
Software Development: Mary Durnwald and Bob Carroll
Executive Marketing Manager: Jeff Weidenaar
Marketing Assistant: Kendra Bassi
Senior Author Support/Technology Specialist: Joe Vetere
Senior Prepress Supervisor: Caroline Fell
Manufacturing Manager: Evelyn Beaton
Production Coordinator: Kathy Diamond
Composition: Nesbitt Graphics, Inc.
Illustrations: Karen Heyt, IllustraTech
Cover Design: Rokusek Design

Cover image: Forest Edge, Hokuto, Hokkaido, Japan 2004 © Michael Kenna

About the cover: The cover image of a tree line on a snow-swept landscape, by the photographer Michael Kenna, was taken in Hokkaido, Japan. The artist was not thinking of calculus when he composed the image, but rather, of a visual haiku consisting of a few elements that would spark the viewer's imagination. Similarly, the minimal design of this text allows the central ideas of calculus developed in this book to unfold to ignite the learner's imagination.

Library of Congress Cataloging-in-Publication Data

Weir, Maurice D.
Thomas' Calculus / Maurice D. Weir, Joel Hass, George B. Thomas.—12th ed.
p. cm
ISBN 978-0-321-58799-2
1. Calculus—Textbooks. I. Hass, Joel. II. Thomas, George B. (George Brinton), 1914–2006. III. Thomas, George B. (George Brinton), 1914–2006. Calculus. IV. Title V. Title: Calculus.

QA303.2.W45 2009b
515–dc22 2009023069

1 2 3 4 5 6 7 8 9 10—CRK—12 11 10 09

Addison-Wesley
is an imprint of

PEARSON

www.pearsoned.com

ISBN-10: 0-321-58799-5
ISBN-13: 978-0-321-58799-2

Contents

14 Partial Derivatives 747

15 Multiple Integrals 836

16 Integration in Vector Fields 901

17 Second-Order Differential Equations online

Appendices AP-1

PREFACE

We have significantly revised this edition of *Thomas' Calculus* to meet the changing needs of today's instructors and students. The result is a book with more examples, more mid-level exercises, more figures, better conceptual flow, and increased clarity and precision. As with previous editions, this new edition provides a modern introduction to calculus that supports conceptual understanding but retains the essential elements of a traditional course. These enhancements are closely tied to an expanded version for this text of MyMathLab® (discussed further on), providing additional support for students and flexibility for instructors.

Many of our students were exposed to the terminology and computational aspects of calculus during high school. Despite this familiarity, students' algebra and trigonometry skills often hinder their success in the college calculus sequence. With this text, we have sought to balance the students' prior experience with calculus with the algebraic skill development they may still need, all without undermining or derailing their confidence. We have taken care to provide enough review material, fully stepped-out solutions, and exercises to support complete understanding for students of all levels.

We encourage students to think beyond memorizing formulas and to generalize concepts as they are introduced. Our hope is that after taking calculus, students will be confident in their problem-solving and reasoning abilities. Mastering a beautiful subject with practical applications to the world is its own reward, but the real gift is the ability to think and generalize. We intend this book to provide support and encouragement for both.

Changes for the Twelfth Edition

CONTENT In preparing this edition we have maintained the basic structure of the Table of Contents from the eleventh edition. Yet we have paid attention to requests by current users and reviewers to postpone the introduction of parametric equations until we present polar coordinates, and to treat l'Hôpital's Rule after the transcendental functions have been studied. We have made numerous revisions to most of the chapters, detailed as follows.

- **Functions** We condensed this chapter even more to focus on reviewing function concepts. Prerequisite material covering real numbers, intervals, increments, straight lines, distances, circles, and parabolas is presented in Appendices 1–3.
- **Limits** To improve the flow of this chapter, we combined the ideas of limits involving infinity and their associations with asymptotes to the graphs of functions, placing them together in the final chapter section.
- **Differentiation** While we use rates of change and tangents to curves as motivation for studying the limit concept, we now merge the derivative concept into a single chapter. We reorganized and increased the number of related rates examples, and we added new examples and exercises on graphing rational functions.

- **Antiderivatives and Integration** We maintain the organization of the eleventh edition in placing antiderivatives as the final topic of the chapter covering applications of derivatives. Our focus is on "recovering a function from its derivative" as the solution to the simplest type of first-order differential equation. Integrals, as "limits of Riemann sums," motivated primarily by the problem of finding the areas of general regions with curved boundaries, are a new topic forming the substance of Chapter 5. After carefully developing the integral *concept*, we turn our attention to its evaluation and connection to antiderivatives captured in the Fundamental Theorem of Calculus. The ensuing applications then *define* the various geometric ideas of area, volume, lengths of paths, and centroids all as limits of Riemann sums giving definite integrals, which can be evaluated by finding an antiderivative of the integrand. We return later to the topic of solving more complicated first-order differential equations, after we define and establish the transcendental functions and their properties.
- **Differential Equations** Some universities prefer that this subject be treated in a course separate from calculus. Although we do cover solutions to separable differential equations when treating exponential growth and decay applications in the chapter on transcendental functions, we organize the bulk of our material into two chapters (which may be omitted for the calculus sequence). We give an introductory treatment of first-order differential equations in Chapter 9, including a new section on systems and phase planes, with applications to the competitive-hunter and predator-prey models. We present an introduction to second-order differential equations in Chapter 17, which is included in MyMathLab as well as the *Thomas' Calculus* Web site, **www.pearsonhighered.com/thomas**.
- **Series** We retain the organizational structure and content of the eleventh edition for the topics of sequences and series. We have added several new figures and exercises to the various sections, and we revised some of the proofs related to convergence of power series in order to improve the accessibility of the material for students. The request stated by one of our users as, "anything you can do to make this material easier for students will be welcomed by our faculty," drove our thinking for revisions to this chapter.
- **Parametric Equations** Several users requested that we move this topic into Chapter 11, where we also cover polar coordinates and conic sections. We have done this, realizing that many departments choose to cover these topics at the beginning of Calculus III, in preparation for their coverage of vectors and multivariable calculus.
- **Vector-Valued Functions** We streamlined the topics in this chapter to place more emphasis on the conceptual ideas supporting the later material on partial derivatives, the gradient vector, and line integrals. We condensed the discussions of the Frenet frame and Kepler's three laws of planetary motion.
- **Multivariable Calculus** We have further enhanced the art in these chapters, and we have added many new figures, examples, and exercises. We reorganized the opening material on double integrals, and combined the applications of double and triple integrals to masses and moments into a single section covering both two- and three-dimensional cases. This reorganization allows for better flow of the key mathematical concepts, together with their properties and computational aspects. As with the eleventh edition, we continue to make the connections of multivariable ideas with their single-variable analogues studied earlier in the book.
- **Vector Fields** We devoted considerable effort to improving the clarity and mathematical precision of our treatment of vector integral calculus, including many additional examples, figures, and exercises. Important theorems and results are stated more clearly and completely, together with enhanced explanations of their hypotheses and mathematical consequences. The area of a surface is now organized into a single section, and surfaces defined implicitly or explicitly are treated as special cases of the more general parametric representation. Surface integrals and their applications then follow as a separate section. Stokes' Theorem and the Divergence Theorem are still presented as generalizations of Green's Theorem to three dimensions.

EXERCISES AND EXAMPLES We know that the exercises and examples are critical components in learning calculus. Because of this importance, we have updated, improved, and increased the number of exercises in nearly every section of the book. There are over 700 new exercises in this edition. We continue our organization and grouping of exercises by topic as in earlier editions, progressing from computational problems to applied and theoretical problems. Exercises requiring the use of computer software systems (such as *Maple*® or *Mathematica*®) are placed at the end of each exercise section, labeled **Computer Explorations**. Most of the applied exercises have a subheading to indicate the kind of application addressed in the problem.

Many sections include new examples to clarify or deepen the meaning of the topic being discussed, and to help students understand its mathematical consequences or applications to science and engineering. At the same time, we have removed examples that were a repetition of material already presented.

ART Because of their importance to learning calculus, we have continued to improve existing figures in *Thomas' Calculus* and we have created a significant number of new ones. We continue to use color consistently and pedagogically to enhance the conceptual idea that is being illustrated. We have also taken a fresh look at all of the figure captions, paying considerable attention to clarity and precision in short statements.

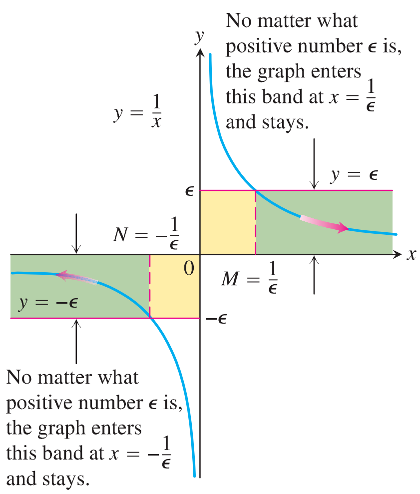


FIGURE 2.50, page 85 The geometric explanation of a finite limit as $x \to \pm\infty$.

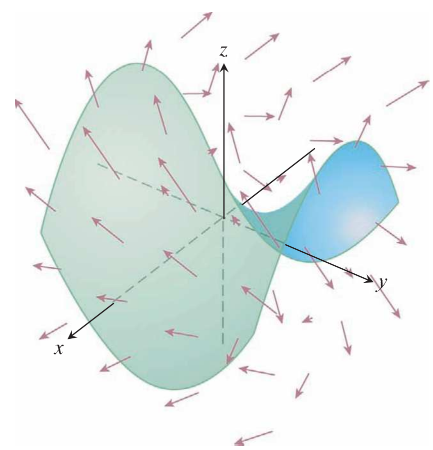


FIGURE 16.9, page 908 A surface in a space occupied by a moving fluid.

MYMATHLAB AND MATHXL The increasing use of and demand for online homework systems has driven the changes to MyMathLab and MathXL® for *Thomas' Calculus*. The **MyMathLab** course now includes significantly more exercises of all types. New Java™ applets add to the already significant collection to help students visualize the concepts and generalize the material.

Continuing Features

RIGOR The level of rigor is consistent with that of earlier editions. We continue to distinguish between formal and informal discussions, and to point out their differences. We think starting with a more intuitive, less formal approach helps students understand a new or difficult concept so they can then appreciate its full mathematical precision and outcomes. We pay attention to defining ideas carefully and to proving theorems appropriate for

calculus students, while mentioning deeper or subtler issues they would study in a more advanced course. Our organization, and distinctions between informal and formal discussions, gives the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove the Intermediate Value Theorem or the Extreme Value Theorem for continuous functions on $a \le x \le b$, we do state these theorems precisely, illustrate their meanings in numerous examples, and use them to prove other important results. Furthermore, for those instructors who desire greater depth of coverage, we discuss in Appendix 6 the reliance of the validity of these theorems on the completeness of the real numbers.

WRITING EXERCISES Writing exercises placed throughout the text ask students to explore and explain a variety of calculus concepts and applications. In addition, the end of each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises make good writing assignments.

END-OF-CHAPTER REVIEWS AND PROJECTS In addition to problems appearing after each section, each chapter culminates with review questions, practice exercises covering the entire chapter, and a series of Additional and Advanced Exercises serving to include more challenging or synthesizing problems. Most chapters also include descriptions of several **Technology Application Projects** that can be worked by individual students, or groups of students, over a longer period of time. These projects require the use of a computer, running *Mathematica* or *Maple*, and additional material that is available over the Internet at **www.pearsonhighered.com/thomas** and in MyMathLab.

WRITING AND APPLICATIONS As always, this text continues to be easy to read, conversational, and mathematically rich. Each new topic is motivated by clear, easy-to-understand examples and is then reinforced by its application to real-world problems of immediate interest to students. A hallmark of this book has been the application of calculus to science and engineering. These applied problems have been updated, improved, and extended continually over the last several editions.

TECHNOLOGY In a course using the text, technology can be incorporated according to the taste of the instructor. Each section contains exercises requiring the use of technology; these are marked with a T if suitable for calculator or computer use or are labeled **Computer Explorations** if a computer algebra system (CAS, such as *Maple* or *Mathematica*) is required.

Text Versions

THOMAS' CALCULUS, Twelfth Edition

Complete (Chapters 1–16), ISBN 0-321-58799-5 | 978-0-321-58799-2
Single Variable Calculus (Chapters 1–11), ISBN 0-321-63742-9 | 978-0-321-63742-0
Multivariable Calculus (Chapters 11–16), ISBN 0-321-64369-0 | 978-0-321-64369-8

THOMAS' CALCULUS: EARLY TRANSCENDENTALS, Twelfth Edition

Complete (Chapters 1–16), ISBN 0-321-58876-2 | 978-0-321-58876-0
Single Variable Calculus (Chapters 1–11), 0-321-62883-7 | 978-0-321-62883-1
Multivariable Calculus (Chapters 11–16), ISBN 0-321-64369-0 | 978-0-321-64369-8

The early transcendentals version of *Thomas' Calculus* introduces and integrates transcendental functions (such as inverse trigonometric, exponential, and logarithmic functions) into the exposition, examples, and exercises of the early chapters alongside the algebraic functions. The Multivariable book for *Thomas' Calculus: Early Transcendentals* is the same text as *Thomas' Calculus, Multivariable*.

Instructor's Editions

Thomas' Calculus, ISBN 0-321-60075-4 | 978-0-321-60075-2
Thomas' Calculus: Early Transcendentals, ISBN 0-321-62718-0 | 978-0-321-62718-6
In addition to including all of the answers present in the student editions, the *Instructor's Editions* include even-numbered answers for Chapters 1–6.

University Calculus (Early Transcendentals)
University Calculus: Alternative Edition (Late Transcendentals)
University Calculus: Elements with Early Transcendentals

The *University Calculus* texts are based on *Thomas' Calculus* and feature a streamlined presentation of the contents of the calculus course. For more information about these titles, visit **www.pearsonhighered.com**.

Print Supplements

INSTRUCTOR'S SOLUTIONS MANUAL

Single Variable Calculus (Chapters 1–11), ISBN 0-321-60807-0 | 978-0-321-60807-9
Multivariable Calculus (Chapters 11–16), ISBN 0-321-60072-X | 978-0-321-60072-1
The *Instructor's Solutions Manual* by William Ardis, Collin County Community College, contains complete worked-out solutions to all of the exercises in the text.

STUDENT'S SOLUTIONS MANUAL

Single Variable Calculus (Chapters 1–11), ISBN 0-321-60070-3 | 978-0-321-60070-7
Multivariable Calculus (Chapters 11–16), ISBN 0-321-60071-1 | 978-0-321-60071-4
The *Student's Solutions Manual* by William Ardis, Collin County Community College, is designed for the student and contains carefully worked-out solutions to all the odd-numbered exercises in the text.

JUST-IN-TIME ALGEBRA AND TRIGONOMETRY FOR CALCULUS, Fourth Edition

ISBN 0-321-67104-X | 978-0-321-67104-2
Sharp algebra and trigonometry skills are critical to mastering calculus, and *Just-in-Time Algebra and Trigonometry for Calculus* by Guntram Mueller and Ronald I. Brent is designed to bolster these skills while students study calculus. As students make their way through calculus, this text is with them every step of the way, showing them the necessary algebra or trigonometry topics and pointing out potential problem spots. The easy-to-use table of contents has algebra and trigonometry topics arranged in the order in which students will need them as they study calculus.

CALCULUS REVIEW CARDS

The Calculus Review Cards (one for Single Variable and another for Multivariable) are a student resource containing important formulas, functions, definitions, and theorems that correspond precisely to *Thomas' Calculus*. These cards can work as a reference for completing homework assignments or as an aid in studying, and are available bundled with a new text. Contact your Pearson sales representative for more information.

Media and Online Supplements

TECHNOLOGY RESOURCE MANUALS

Maple Manual by James Stapleton, North Carolina State University
Mathematica Manual by Marie Vanisko, Carroll College
TI-Graphing Calculator Manual by Elaine McDonald-Newman, Sonoma State University
These manuals cover *Maple* 13, *Mathematica* 7, and the TI-83 Plus/TI-84 Plus and TI-89, respectively. Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, including syntax and commands.

These manuals are available to qualified instructors through the Pearson Instructor Resource Center, **www.pearsonhighered/irc**, and MyMathLab.

WEB SITE www.pearsonhighered.com/thomas

The *Thomas' Calculus* Web site contains the chapter on Second-Order Differential Equations, including odd-numbered answers, and provides the expanded historical biographies and essays referenced in the text. Also available is a collection of *Maple* and *Mathematica* modules, as well as the **Technology Application Projects**, which can be used as projects by individual students or groups of students.

MyMathLab Online Course (access code required)

MyMathLab is a text-specific, easily customizable online course that integrates interactive multimedia instruction with textbook content. MyMathLab gives you the tools you need to deliver all or a portion of your course online, whether your students are in a lab setting or working from home.

- **Interactive homework exercises**, correlated to your textbook at the objective level, are algorithmically generated for unlimited practice and mastery. Most exercises are free-response and provide guided solutions, sample problems, and learning aids for extra help.
- **"Getting Ready" chapter** includes hundreds of exercises that address prerequisite skills in algebra and trigonometry. Each student can receive remediation for just those skills he or she needs help with.
- **Personalized Study Plan**, generated when students complete a test or quiz, indicates which topics have been mastered and links to tutorial exercises for topics students have not mastered.
- **Multimedia learning aids**, such as video lectures, Java applets, animations, and a complete multimedia textbook, help students independently improve their understanding and performance.
- **Assessment Manager** lets you create online homework, quizzes, and tests that are automatically graded. Select just the right mix of questions from the MyMathLab exercise bank and instructor-created custom exercises.
- **Gradebook**, designed specifically for mathematics and statistics, automatically tracks students' results and gives you control over how to calculate final grades. You can also add offline (paper-and-pencil) grades to the gradebook.
- **MathXL Exercise Builder** allows you to create static and algorithmic exercises for your online assignments. You can use the library of sample exercises as an easy starting point.
- **Pearson Tutor Center** (**www.pearsontutorservices.com**) access is automatically included with MyMathLab. The Tutor Center is staffed by qualified math instructors who provide textbook-specific tutoring for students via toll-free phone, fax, email, and interactive Web sessions.

MyMathLab is powered by CourseCompass™, Pearson Education's online teaching and learning environment, and by MathXL, our online homework, tutorial, and assessment system. MyMathLab is available to qualified adopters. For more information, visit **www.mymathlab.com** or contact your Pearson sales representative.

Video Lectures with Optional Captioning

The Video Lectures with Optional Captioning feature an engaging team of mathematics instructors who present comprehensive coverage of topics in the text. The lecturers' presentations include examples and exercises from the text and support an approach that emphasizes visualization and problem solving. Available only through MyMathLab and MathXL.

MathXL Online Course (access code required)

MathXL is an online homework, tutorial, and assessment system that accompanies Pearson's textbooks in mathematics or statistics.

- **Interactive homework exercises**, correlated to your textbook at the objective level, are algorithmically generated for unlimited practice and mastery. Most exercises are free-response and provide guided solutions, sample problems, and learning aids for extra help.
- **"Getting Ready" chapter** includes hundreds of exercises that address prerequisite skills in algebra and trigonometry. Each student can receive remediation for just those skills he or she needs help with.
- **Personalized Study Plan,** generated when students complete a test or quiz, indicates which topics have been mastered and links to tutorial exercises for topics students have not mastered.
- **Multimedia learning aids**, such as video lectures, Java applets, and animations, help students independently improve their understanding and performance.
- **Gradebook,** designed specifically for mathematics and statistics, automatically tracks students' results and gives you control over how to calculate final grades.
- **MathXL Exercise Builder** allows you to create static and algorithmic exercises for your online assignments. You can use the library of sample exercises as an easy starting point.
- **Assessment Manager** lets you create online homework, quizzes, and tests that are automatically graded. Select just the right mix of questions from the MathXL exercise bank, or instructor-created custom exercises.

MathXL is available to qualified adopters. For more information, visit our Web site at **www.mathxl.com**, or contact your Pearson sales representative.

TestGen®

TestGen (**www.pearsonhighered.com/testgen**) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions. Tests can be printed or administered online. The software and testbank are available for download from Pearson Education's online catalog.

PowerPoint® Lecture Slides

These classroom presentation slides are geared specifically to the sequence and philosophy of *Thomas' Calculus* series. Key graphics from the book are included to help bring the concepts alive in the classroom.These files are available to qualified instructors through the Pearson Instructor Resource Center, **www.pearsonhighered/irc**, and MyMathLab.

Acknowledgments

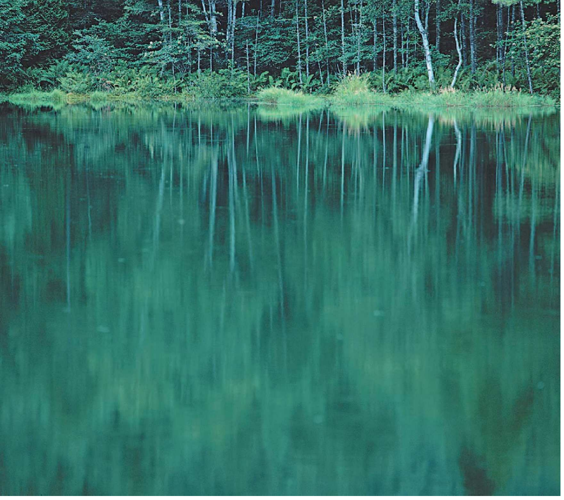
We would like to express our thanks to the people who made many valuable contributions to this edition as it developed through its various stages:

Accuracy Checkers

Blaise DeSesa
Paul Lorczak
Kathleen Pellissier
Lauri Semarne
Sarah Streett
Holly Zullo

Reviewers for the Twelfth Edition

Meighan Dillon, *Southern Polytechnic State University*
Anne Dougherty, *University of Colorado*
Said Fariabi, *San Antonio College*
Klaus Fischer, *George Mason University*
Tim Flood, *Pittsburg State University*
Rick Ford, *California State University—Chico*
Robert Gardner, *East Tennessee State University*
Christopher Heil, *Georgia Institute of Technology*
Joshua Brandon Holden, *Rose-Hulman Institute of Technology*
Alexander Hulpke, *Colorado State University*
Jacqueline Jensen, *Sam Houston State University*
Jennifer M. Johnson, *Princeton University*
Hideaki Kaneko, *Old Dominion University*
Przemo Kranz, *University of Mississippi*
Xin Li, *University of Central Florida*
Maura Mast, *University of Massachusetts—Boston*
Val Mohanakumar, *Hillsborough Community College—Dale Mabry Campus*
Aaron Montgomery, *Central Washington University*
Cynthia Piez, *University of Idaho*
Brooke Quinlan, *Hillsborough Community College—Dale Mabry Campus*
Rebecca A. Segal, *Virginia Commonwealth University*
Andrew V. Sills, *Georgia Southern University*
Alex Smith, *University of Wisconsin—Eau Claire*
Mark A. Smith, *Miami University*
Donald Solomon, *University of Wisconsin—Milwaukee*
Blake Thornton, *Washington University in St. Louis*
David Walnut, *George Mason University*
Adrian Wilson, *University of Montevallo*
Bobby Winters, *Pittsburg State University*
Dennis Wortman, *University of Massachusetts—Boston*



1 FUNCTIONS

OVERVIEW Functions are fundamental to the study of calculus. In this chapter we review what functions are and how they are pictured as graphs, how they are combined and transformed, and ways they can be classified. We review the trigonometric functions, and we discuss misrepresentations that can occur when using calculators and computers to obtain a function's graph. The real number system, Cartesian coordinates, straight lines, parabolas, and circles are reviewed in the Appendices. We treat inverse, exponential, and logarithmic functions in Chapter 7.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these function ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels at constant speed along a straight-line path depends on the elapsed time.

In each case, the value of one variable quantity, say y, depends on the value of another variable quantity, which we might call x. We say that "y is a function of x" and write this symbolically as

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

In this notation, the symbol f represents the function, the letter x is the **independent variable** representing the input value of f, and y is the **dependent variable** or output value of f at x.

> **DEFINITION** A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 13–16, we will encounter functions for which the elements of the sets are points in the coordinate plane or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r (so r, interpreted as a length, can only be positive in this formula). When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix 1), the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

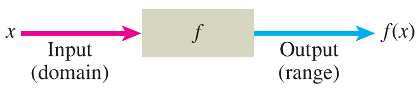


FIGURE 1.1 A diagram showing a function as a kind of machine.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite. The range of a function is not always easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the $\sqrt{x}$ key.

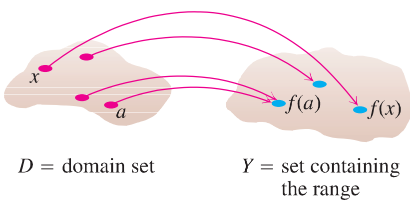


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates an element of the domain D with a unique or single element in the set Y. In Figure 1.2, the arrows indicate that $f(a)$ is associated with a, $f(x)$ is associated with x, and so on. Notice that a function can have the same *value* at two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

EXAMPLE 1 Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above the point x. The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

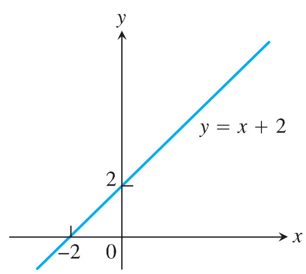


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

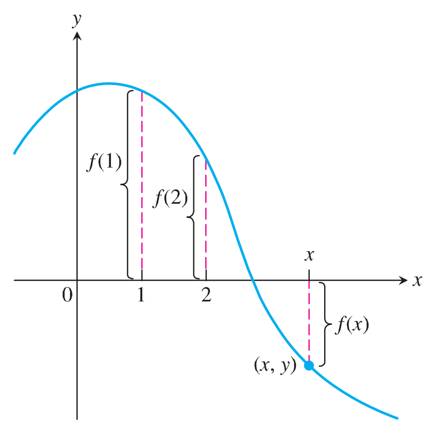


FIGURE 1.4 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

Solution Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

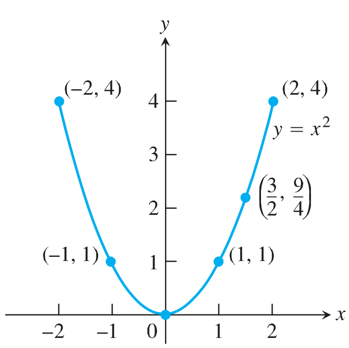
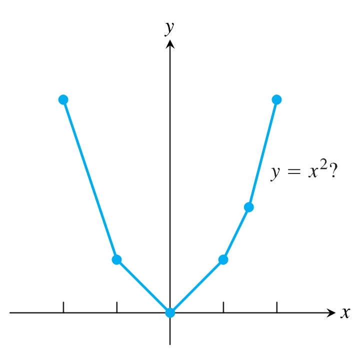
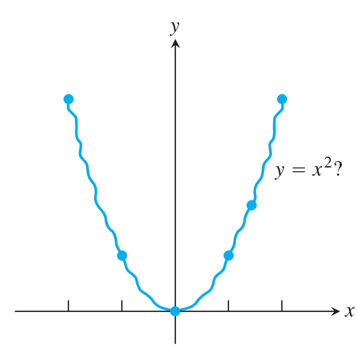


FIGURE 1.5 Graph of the function in Example 2.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?





To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile we will have to settle for plotting points and connecting them as best we can.

Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula (the area function) and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. Numerical representations are often used by engineers and scientists. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data in Table 1.1 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function over time. If we first make a scatterplot and then connect approximately the data points (t, p) from the table, we obtain the graph shown in Figure 1.6.

TABLE 1.1 Tuning fork data

Time	Pressure	Time	Pressure
0.00091	−0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	−0.164
0.00271	−0.141	0.00543	−0.320
0.00289	−0.309	0.00562	−0.354
0.00307	−0.348	0.00579	−0.248
0.00325	−0.248	0.00598	−0.035
0.00344	−0.041		

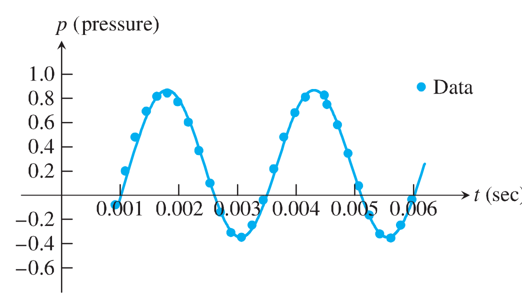


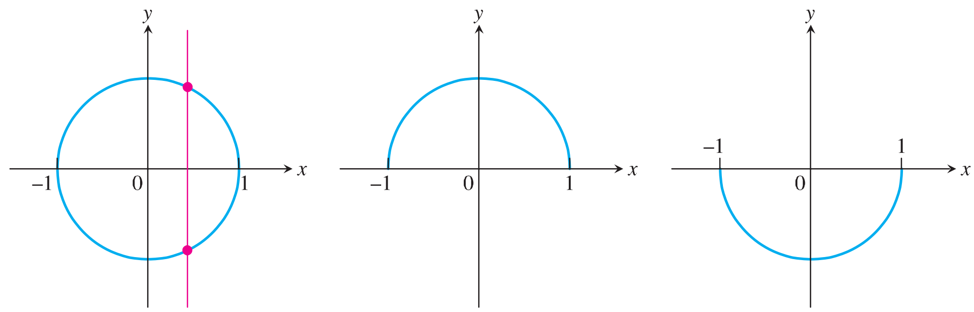
FIGURE 1.6 A smooth curve through the plotted points gives a graph of the pressure function represented by Table 1.1 (Example 3).

■

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function since some vertical lines intersect the circle twice. The circle in Figure 1.7a, however, does contain the graphs of *two* functions of x: the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

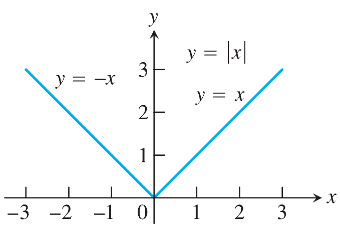


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

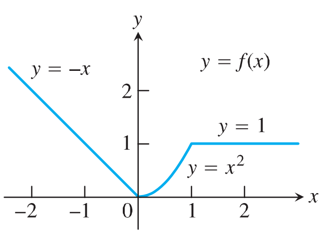


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

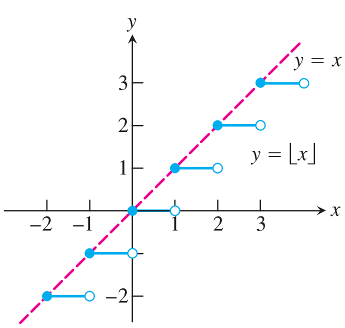


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9). ■

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$ ■

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges \$1 for each hour or part of an hour. ■

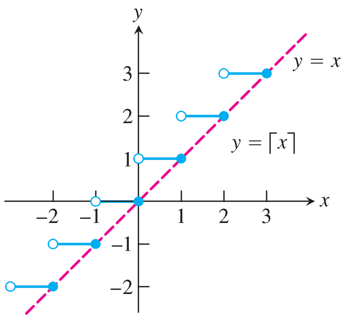


FIGURE 1.11 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

Increasing and Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

> **DEFINITIONS** Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.
>
> 1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
> 2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly* increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded) and by definition never consists of a single point (Appendix 1).

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

> **DEFINITIONS** A function $y = f(x)$ is an
>
> **even function of x** if $f(-x) = f(x)$,
> **odd function of x** if $f(-x) = -f(x)$,
>
> for every x in the function's domain.

The names *even* and *odd* come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f.

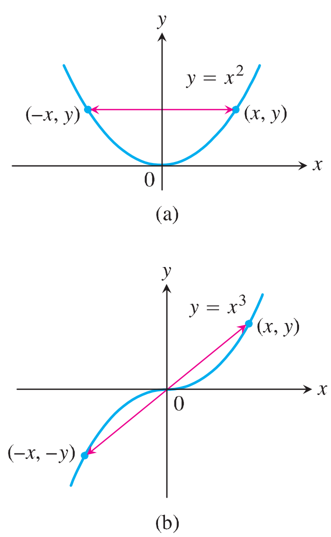


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

EXAMPLE 8

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.13a).
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

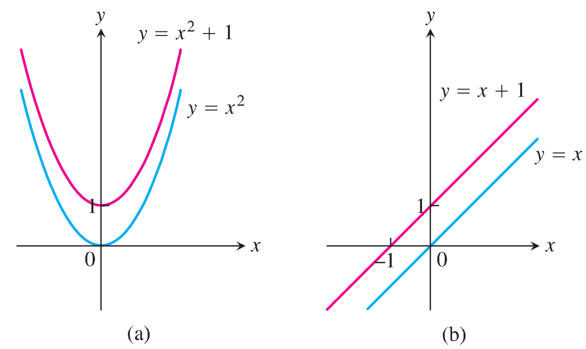


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 8).

Common Functions

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope $m = 0$ (Figure 1.14b). A linear function with positive slope whose graph passes through the origin is called a *proportionality* relationship.

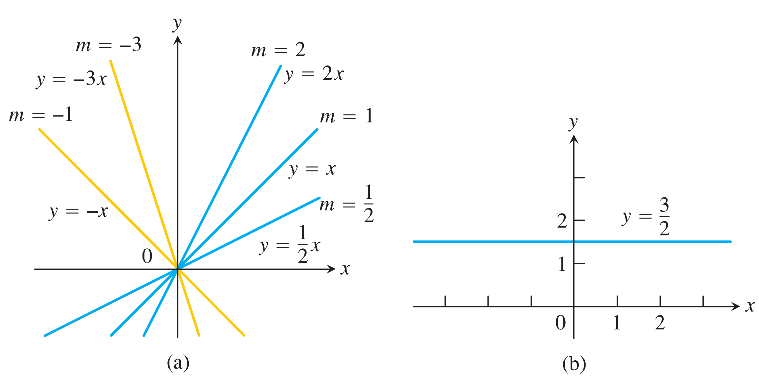


FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point (1, 1) and through the origin. The graphs of functions with even powers are symmetric about the y-axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

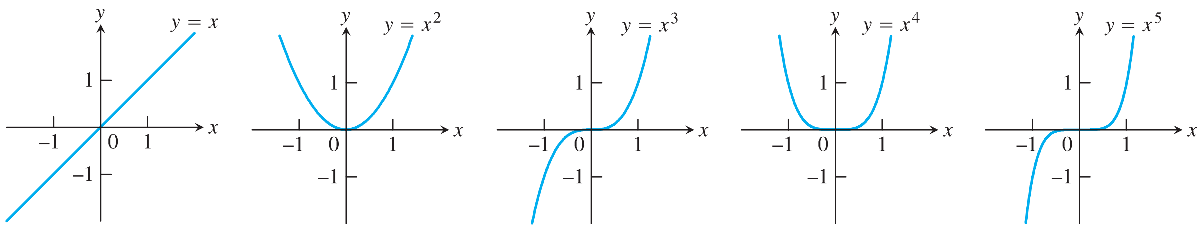


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y-axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

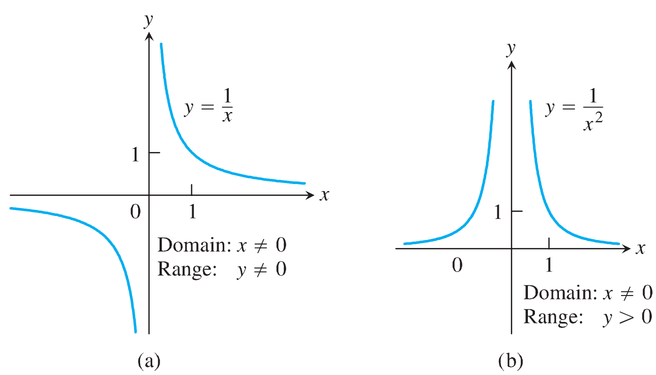


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.17 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the

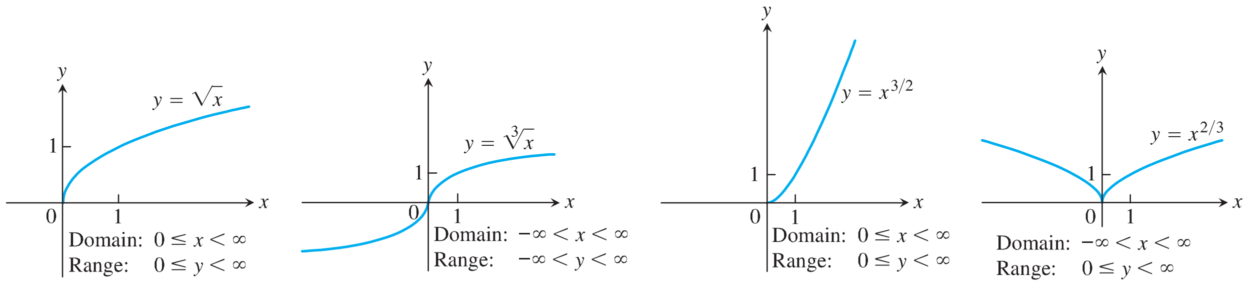


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.18 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

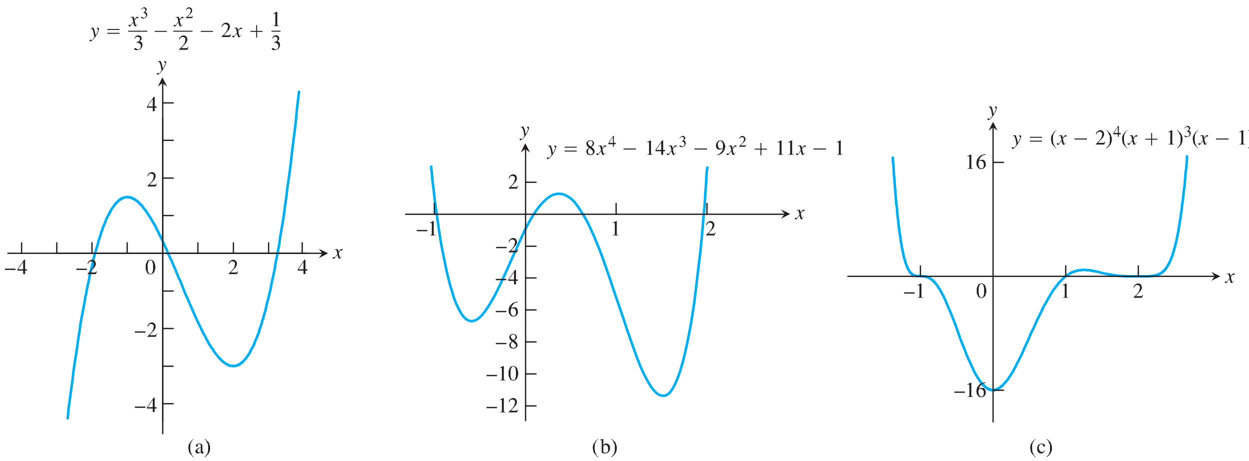


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.19.

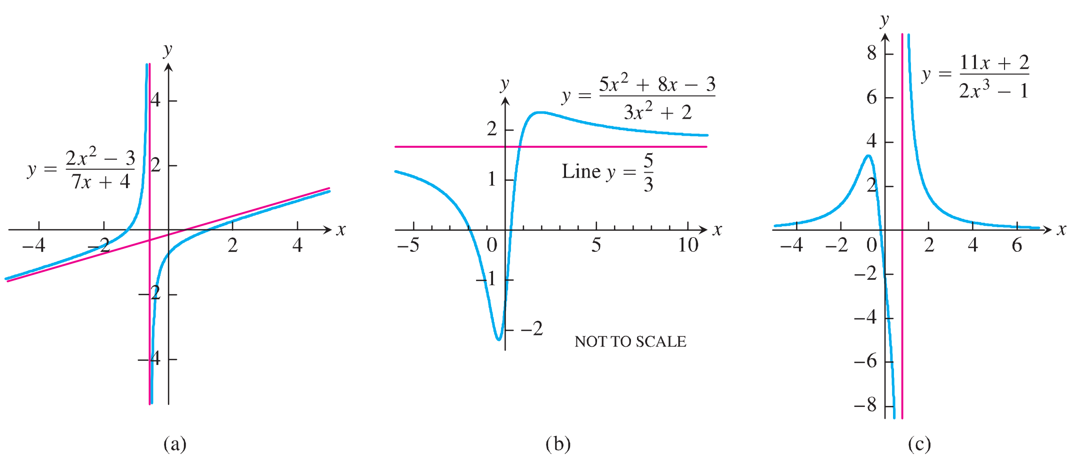


FIGURE 1.19 Graphs of three rational functions. The straight red lines are called *asymptotes* and are not part of the graph.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.20 displays the graphs of three algebraic functions.

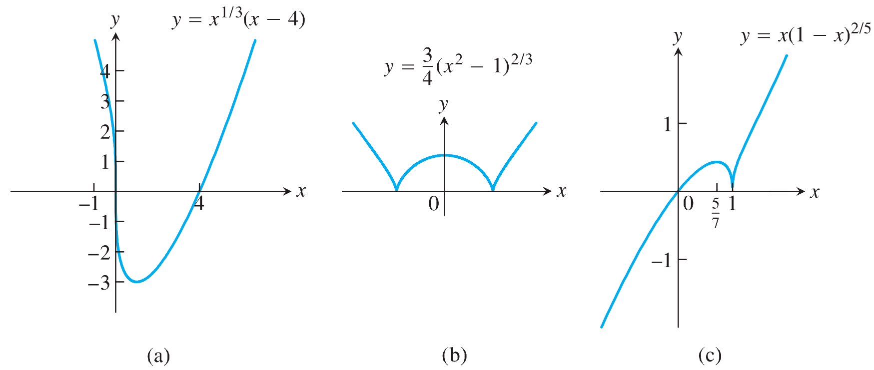


FIGURE 1.20 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

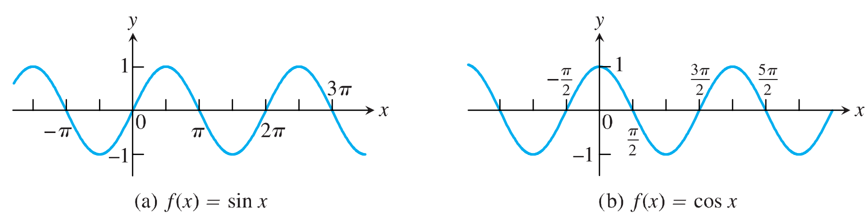


FIGURE 1.21 Graphs of the sine and cosine functions.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. We study exponential functions in Section 7.3. The graphs of some exponential functions are shown in Figure 1.22.

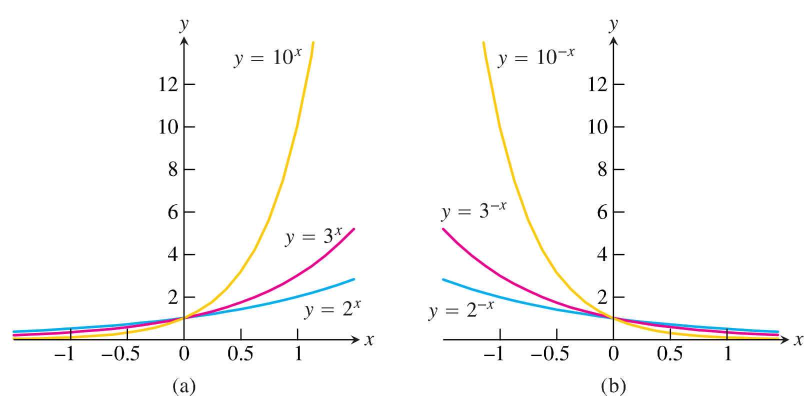


FIGURE 1.22 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and the calculus of these functions is studied in Chapter 7. Figure 1.23 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

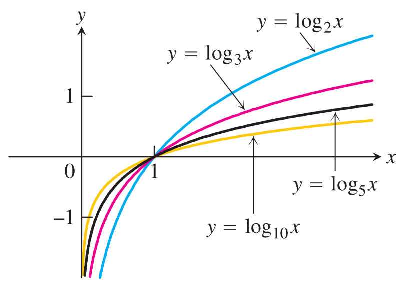


FIGURE 1.23 Graphs of four logarithmic functions.

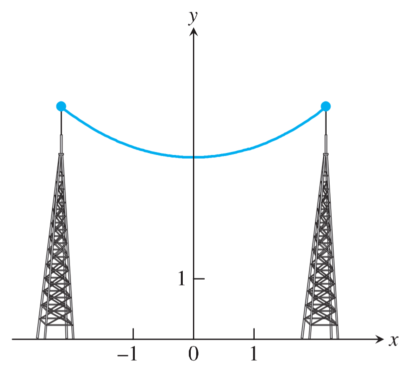


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means "chain.")

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a **catenary**. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.24). The function defining the graph is discussed in Section 7.7.

Exercises 1.1

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(x) = \sqrt{5x + 10}$

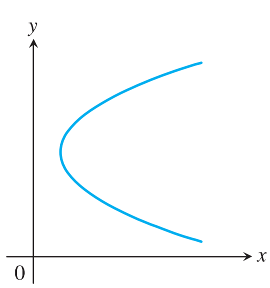
4. $g(x) = \sqrt{x^2 - 3x}$

5. $f(t) = \dfrac{4}{3 - t}$

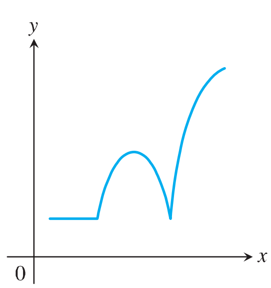
6. $G(t) = \dfrac{2}{t^2 - 16}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

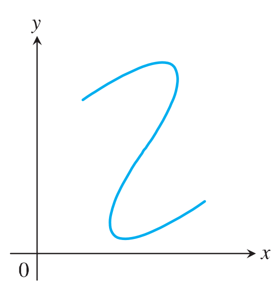
7. a.



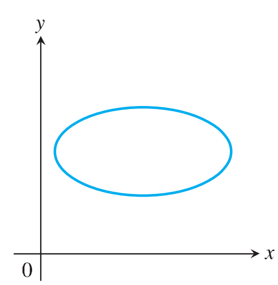
b.



8. a.



b.



Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.

10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.

11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

13. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x.

14. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y.

Functions and Graphs

Find the domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$ 16. $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$ 18. $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$ 20. $G(t) = 1/|t|$

21. Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.

22. Find the range of $y = 2 + \dfrac{x^2}{x^2 + 4}$.

23. Graph the following equations and explain why they are not graphs of functions of x.
   a. $|y| = x$ b. $y^2 = x^2$

24. Graph the following equations and explain why they are not graphs of functions of x.
   a. $|x| + |y| = 1$ b. $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

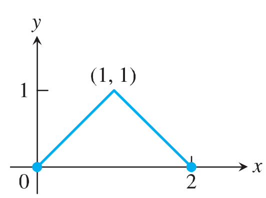
25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

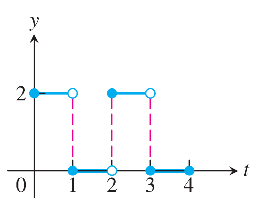
26. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

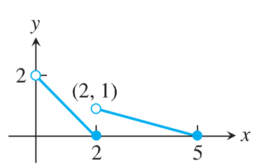
27. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

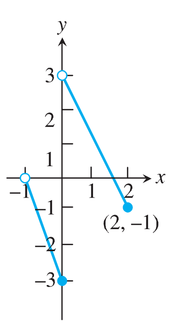
28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

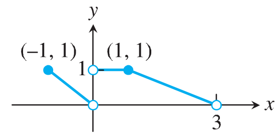
Find a formula for each function graphed in Exercises 29–32.

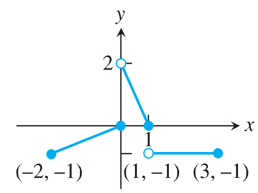
29. a.


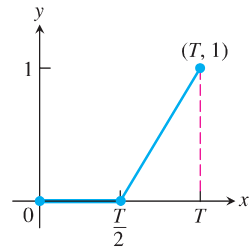
b.


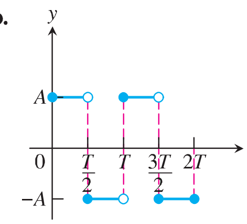
30. a.


b.


31. a.


b.


32. a.


b.


The Greatest and Least Integer Functions

33. For what values of x is
   a. $\lfloor x \rfloor = 0$? b. $\lceil x \rceil = 0$?

34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$ 38. $y = -\dfrac{1}{x^2}$

39. $y = -\dfrac{1}{x}$ 40. $y = \dfrac{1}{|x|}$

41. $y = \sqrt{|x|}$ 42. $y = \sqrt{-x}$

43. $y = x^3/8$ 44. $y = -4\sqrt{x}$

45. $y = -x^{3/2}$ 46. $y = (-x)^{2/3}$

Even and Odd Functions

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$ 48. $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$ 50. $f(x) = x^2 + x$

51. $g(x) = x^3 + x$ 52. $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$ 54. $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$ 56. $h(t) = |t^3|$

57. $h(t) = 2t + 1$ 58. $h(t) = 2|t| + 1$

Theory and Examples

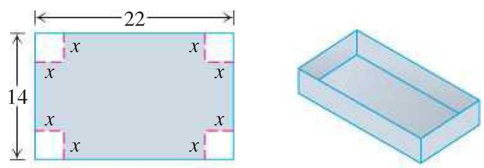
59. The variable s is proportional to t, and $s = 25$ when $t = 75$. Determine t when $s = 60$.

60. Kinetic energy The kinetic energy K of a mass is proportional to the square of its velocity v. If $K = 12{,}960$ joules when $v = 18$ m/sec, what is K when $v = 10$ m/sec?

61. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.

62. Boyle's Law Boyle's Law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7\ \text{lbs/in}^2$ when $V = 1000\ \text{in}^3$, then what is V when $P = 23.4\ \text{lbs/in}^2$?

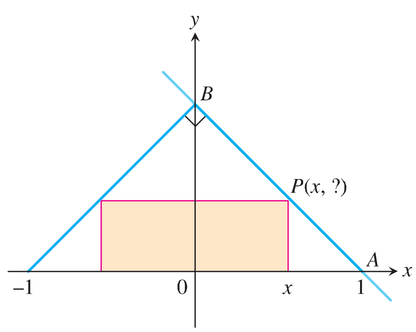
63. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 in. by 22 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



64. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

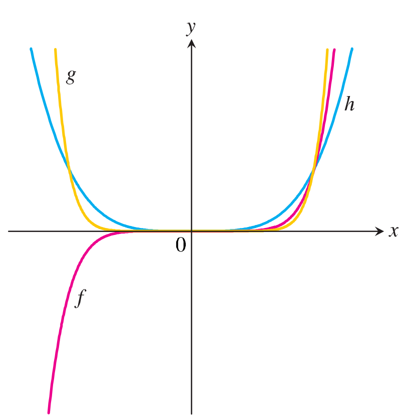
a. Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b. Express the area of the rectangle in terms of x.

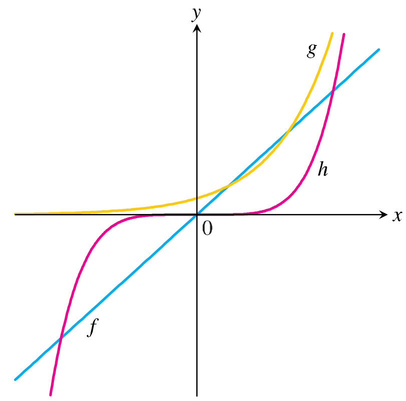


In Exercises 65 and 66, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

65. a. $y = x^4$ **b.** $y = x^7$ **c.** $y = x^{10}$



66. a. $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



T 67. a. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b. Confirm your findings in part (a) algebraically.

T 68. a. Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x-1} < \frac{2}{x+1}.$$

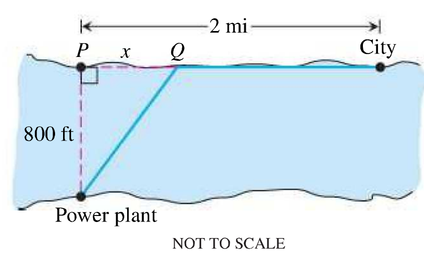
b. Confirm your findings in part (a) algebraically.

69. For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

70. Three hundred books sell for \$40 each, resulting in a revenue of (300)(\$40) = \$12,000. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.

71. A pen in the shape of an isosceles right triangle with legs of length x ft and hypotenuse of length h ft is to be built. If fencing costs \$5/ft for the legs and \$10/ft for the hypotenuse, write the total cost C of construction as a function of h.

72. Industrial costs A power plant sits next to a river where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.



a. Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

b. Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.

1.2 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f+g)(x) = f(x) + g(x).$$
$$(f-g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)g(x).$$

Notice that the + sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1-x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

■

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y-coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.25. The graphs of $f + g$ and $f \cdot g$ from Example 1 are shown in Figure 1.26.

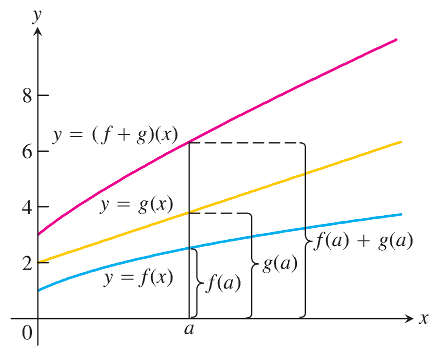


FIGURE 1.25 Graphical addition of two functions.

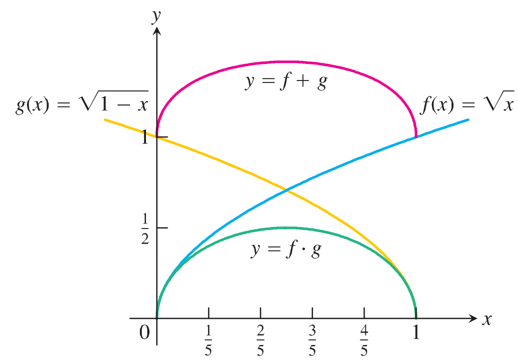


FIGURE 1.26 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite Functions

Composition is another method for combining functions.

> **DEFINITION** If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

The definition implies that $f \circ g$ can be formed when the range of g lies in the domain of f. To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.27 pictures $f \circ g$ as a machine diagram and Figure 1.28 shows the composite as an arrow diagram.

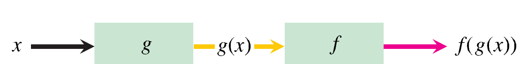


FIGURE 1.27 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

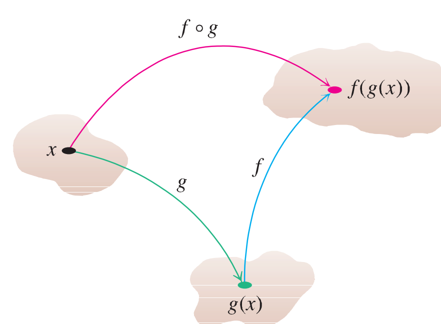


FIGURE 1.28 Arrow diagram for $f \circ g$.

To evaluate the composite function $g \circ f$ (when defined), we find $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

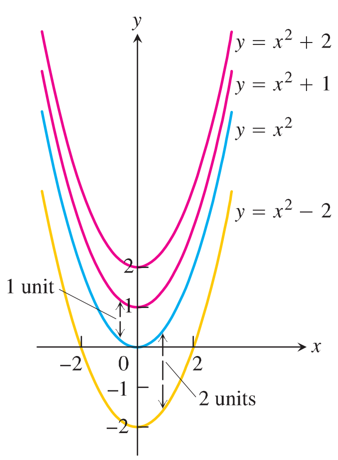


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

EXAMPLE 3

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure 1.30).

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.31). ■

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

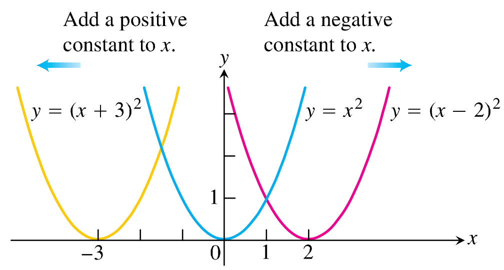


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x.

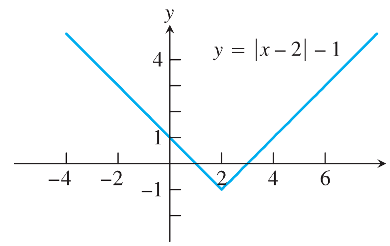


FIGURE 1.31 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 3d).

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x}$.

(a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3 (Figure 1.32).

(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.33). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.34). ■

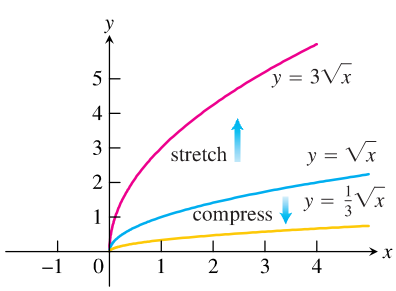


FIGURE 1.32 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4a).

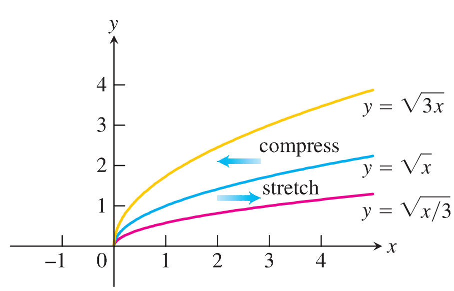


FIGURE 1.33 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4b).

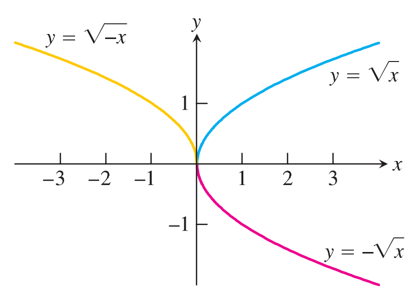


FIGURE 1.34 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 4c).

EXAMPLE 5 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.35a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure 1.35b).

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure 1.35c).

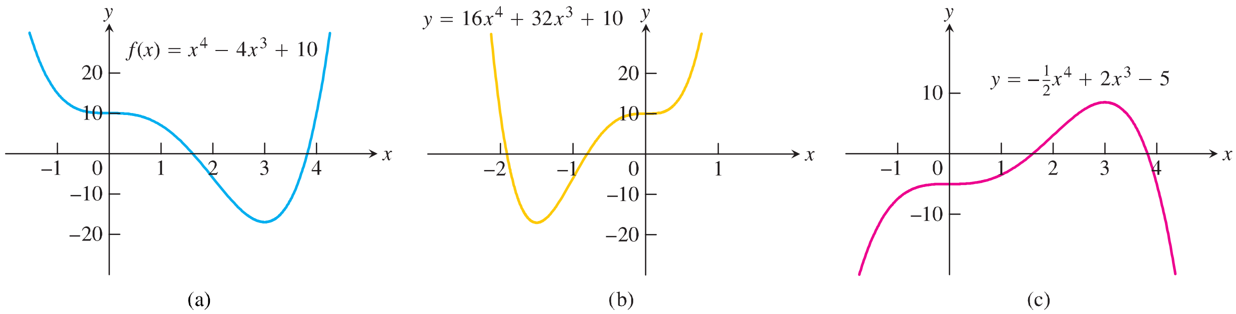


FIGURE 1.35 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 5).

Solution

(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f:

$$\begin{aligned} y = f(-2x) &= (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

(b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$ ■

Ellipses

Although they are not the graphs of functions, circles can be stretched horizontally or vertically in the same way as the graphs of functions. The standard equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Substituting cx for x in the standard equation for a circle (Figure 1.36a) gives

$$c^2x^2 + y^2 = r^2. \tag{1}$$

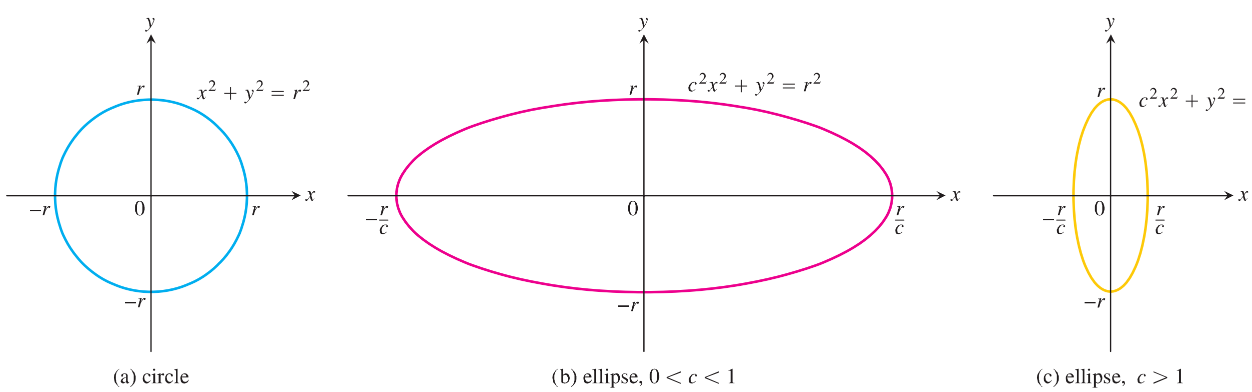


FIGURE 1.36 Horizontal stretching or compression of a circle produces graphs of ellipses.

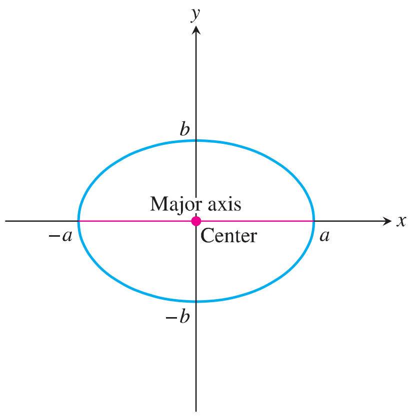


FIGURE 1.37 Graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, where the major axis is horizontal.

If $0 < c < 1$, the graph of Equation (1) horizontally stretches the circle; if $c > 1$ the circle is compressed horizontally. In either case, the graph of Equation (1) is an ellipse (Figure 1.36). Notice in Figure 1.36 that the y-intercepts of all three graphs are always $-r$ and r. In Figure 1.36b, the line segment joining the points $(\pm r/c, 0)$ is called the **major axis** of the ellipse; the **minor axis** is the line segment joining $(0, \pm r)$. The axes of the ellipse are reversed in Figure 1.36c: The major axis is the line segment joining the points $(0, \pm r)$, and the minor axis is the line segment joining the points $(\pm r/c, 0)$. In both cases, the major axis is the longer line segment.

If we divide both sides of Equation (1) by r^2, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2)$$

where $a = r/c$ and $b = r$. If $a > b$, the major axis is horizontal; if $a < b$, the major axis is vertical. The **center** of the ellipse given by Equation (2) is the origin (Figure 1.37).

Substituting $x - h$ for x, and $y - k$ for y, in Equation (2) results in

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \qquad (3)$$

Equation (3) is the **standard equation of an ellipse** with center at (h, k). The geometric definition and properties of ellipses are reviewed in Section 11.6.

Exercises 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$
2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 3 and 4, find the domains and ranges of f, g, f/g, and g/f.

3. $f(x) = 2, \quad g(x) = x^2 + 1$
4. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.
   a. $f(g(0))$ **b.** $g(f(0))$
   c. $f(g(x))$ **d.** $g(f(x))$
   e. $f(f(-5))$ **f.** $g(g(2))$
   g. $f(f(x))$ **h.** $g(g(x))$
6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.
   a. $f(g(1/2))$ **b.** $g(f(1/2))$
   c. $f(g(x))$ **d.** $g(f(x))$
   e. $f(f(2))$ **f.** $g(g(2))$
   g. $f(f(x))$ **h.** $g(g(x))$

In Exercises 7–10, write a formula for $f \circ g \circ h$.

7. $f(x) = x + 1, \quad g(x) = 3x, \quad h(x) = 4 - x$
8. $f(x) = 3x + 4, \quad g(x) = 2x - 1, \quad h(x) = x^2$
9. $f(x) = \sqrt{x + 1}, \quad g(x) = \frac{1}{x + 4}, \quad h(x) = \frac{1}{x}$
10. $f(x) = \frac{x + 2}{3 - x}, \quad g(x) = \frac{x^2}{x^2 + 1}, \quad h(x) = \sqrt{2 - x}$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composite involving one or more of f, g, h, and j.

11. **a.** $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$
    c. $y = x^{1/4}$ **d.** $y = 4x$
    e. $y = \sqrt{(x - 3)^3}$ **f.** $y = (2x - 6)^3$
12. **a.** $y = 2x - 3$ **b.** $y = x^{3/2}$
    c. $y = x^9$ **d.** $y = x - 6$
    e. $y = 2\sqrt{x - 3}$ **f.** $y = \sqrt{x^3 - 3}$
13. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	?
b.	$x + 2$	$3x$	?
c.	?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d.	$\frac{x}{x - 1}$	$\frac{x}{x - 1}$	?
e.	?	$1 + \frac{1}{x}$	x
f.	$\frac{1}{x}$	?	x

14. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\dfrac{1}{x-1}$	$\lvert x \rvert$	?
b.	?	$\dfrac{x-1}{x}$	$\dfrac{x}{x+1}$
c.	?	$\sqrt{x}$	$\lvert x \rvert$
d.	$\sqrt{x}$	?	$\lvert x \rvert$

15. Evaluate each expression using the given table of values

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

a. $f(g(-1))$ **b.** $g(f(0))$ **c.** $f(f(-1))$
d. $g(g(2))$ **e.** $g(f(-2))$ **f.** $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2. \end{cases}$$

a. $f(g(0))$ **b.** $g(f(3))$ **c.** $g(g(-1))$
d. $f(f(2))$ **e.** $g(f(0))$ **f.** $f(g(1/2))$

In Exercises 17 and 18, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

17. $f(x) = \sqrt{x + 1}$, $g(x) = \dfrac{1}{x}$

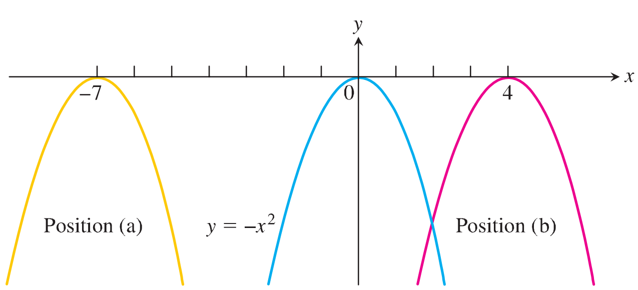
18. $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

19. Let $f(x) = \dfrac{x}{x - 2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

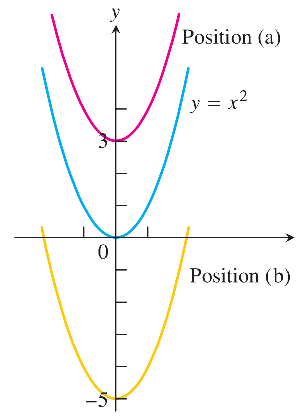
20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

Shifting Graphs

21. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

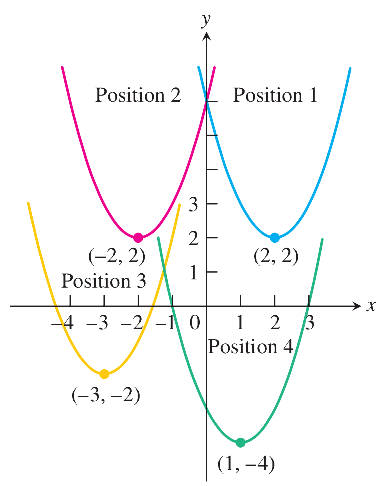


22. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

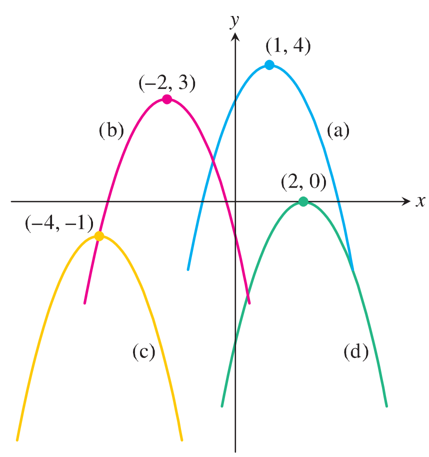


23. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$
c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



24. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 25–34 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

25. $x^2 + y^2 = 49$ Down 3, left 2

26. $x^2 + y^2 = 25$ Up 3, left 4

27. $y = x^3$ Left 1, down 1

28. $y = x^{2/3}$ Right 1, down 1

29. $y = \sqrt{x}$ Left 0.81

30. $y = -\sqrt{x}$ Right 3

31. $y = 2x - 7$ Up 7

32. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

33. $y = 1/x$ Up 1, right 1

34. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 35–54.

35. $y = \sqrt{x + 4}$

36. $y = \sqrt{9 - x}$

37. $y = |x - 2|$

38. $y = |1 - x| - 1$

39. $y = 1 + \sqrt{x - 1}$

40. $y = 1 - \sqrt{x}$

41. $y = (x + 1)^{2/3}$

42. $y = (x - 8)^{2/3}$

43. $y = 1 - x^{2/3}$

44. $y + 4 = x^{2/3}$

45. $y = \sqrt[3]{x - 1} - 1$

46. $y = (x + 2)^{3/2} + 1$

47. $y = \dfrac{1}{x - 2}$

48. $y = \dfrac{1}{x} - 2$

49. $y = \dfrac{1}{x} + 2$

50. $y = \dfrac{1}{x + 2}$

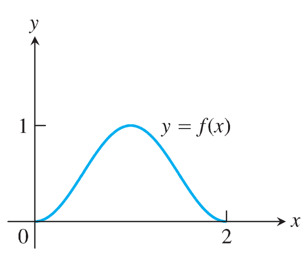
51. $y = \dfrac{1}{(x - 1)^2}$

52. $y = \dfrac{1}{x^2} - 1$

53. $y = \dfrac{1}{x^2} + 1$

54. $y = \dfrac{1}{(x + 1)^2}$

55. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



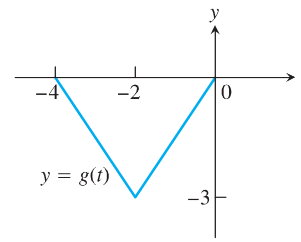
a. $f(x) + 2$ **b.** $f(x) - 1$

c. $2f(x)$ **d.** $-f(x)$

e. $f(x + 2)$ **f.** $f(x - 1)$

g. $f(-x)$ **h.** $-f(x + 1) + 1$

56. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a. $g(-t)$ **b.** $-g(t)$

c. $g(t) + 3$ **d.** $1 - g(t)$

e. $g(-t + 2)$ **f.** $g(t - 2)$

g. $g(1 - t)$ **h.** $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 57–66 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

57. $y = x^2 - 1$, stretched vertically by a factor of 3

58. $y = x^2 - 1$, compressed horizontally by a factor of 2

59. $y = 1 + \dfrac{1}{x^2}$, compressed vertically by a factor of 2

60. $y = 1 + \dfrac{1}{x^2}$, stretched horizontally by a factor of 3

61. $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4

62. $y = \sqrt{x + 1}$, stretched vertically by a factor of 3

63. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2

64. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3

65. $y = 1 - x^3$, compressed horizontally by a factor of 3

66. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 67–74, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

67. $y = -\sqrt{2x + 1}$

68. $y = \sqrt{1 - \dfrac{x}{2}}$

69. $y = (x - 1)^3 + 2$

70. $y = (1 - x)^3 + 2$

71. $y = \dfrac{1}{2x} - 1$

72. $y = \dfrac{2}{x^2} + 1$

73. $y = -\sqrt[3]{x}$

74. $y = (-2x)^{2/3}$

75. Graph the function $y = |x^2 - 1|$.

76. Graph the function $y = \sqrt{|x|}$.

Ellipses

Exercises 77–82 give equations of ellipses. Put each equation in standard form and sketch the ellipse.

77. $9x^2 + 25y^2 = 225$

78. $16x^2 + 7y^2 = 112$

79. $3x^2 + (y - 2)^2 = 3$

80. $(x + 1)^2 + 2y^2 = 4$

81. $3(x - 1)^2 + 2(y + 2)^2 = 6$

82. $6\left(x + \frac{3}{2}\right)^2 + 9\left(y - \frac{1}{2}\right)^2 = 54$

83. Write an equation for the ellipse $(x^2/16) + (y^2/9) = 1$ shifted 4 units to the left and 3 units up. Sketch the ellipse and identify its center and major axis.

84. Write an equation for the ellipse $(x^2/4) + (y^2/25) = 1$ shifted 3 units to the right and 2 units down. Sketch the ellipse and identify its center and major axis.

Combining Functions

85. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

a. fg **b.** f/g **c.** g/f
d. $f^2 = ff$ **e.** $g^2 = gg$ **f.** $f \circ g$
g. $g \circ f$ **h.** $f \circ f$ **i.** $g \circ g$

86. Can a function be both even and odd? Give reasons for your answer.

T **87.** (*Continuation of Example* 1.) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T **88.** Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

Angles

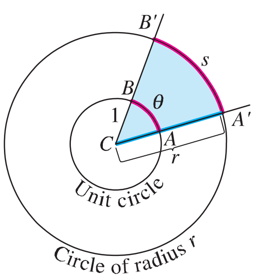


FIGURE 1.38 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of "radius units" contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.38), or

$$\boxed{s = r\theta \qquad (\theta \text{ in radians}). \qquad (1)}$$

If the circle is a unit circle having radius $r = 1$, then from Figure 1.38 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180° \qquad (2)$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

Table 1.2 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.2 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Figure 1.39). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

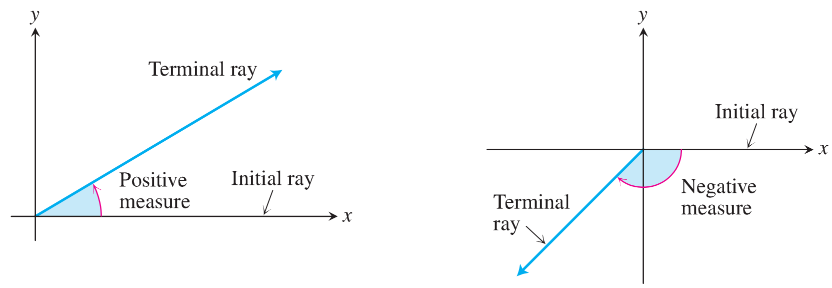


FIGURE 1.39 Angles in standard position in the xy-plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.40).

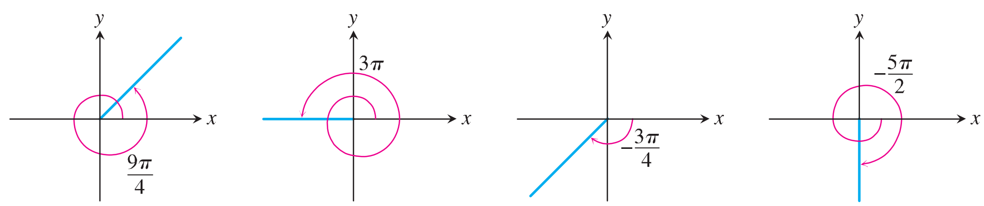
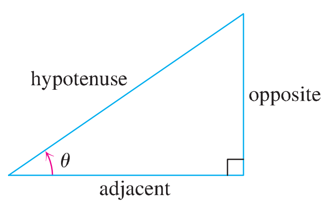


FIGURE 1.40 Nonzero radian measures can be positive or negative and can go beyond 2π.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.41 Trigonometric ratios of an acute angle.

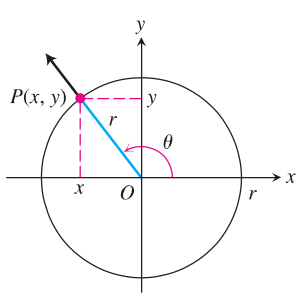


FIGURE 1.42 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

Angle Convention: Use Radians From now on, in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. We use radians because it simplifies many of the operations in calculus, and some results we will obtain involving the trigonometric functions are not true when angles are measured in degrees.

The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.41). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.42).

$$\textbf{sine:} \quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:} \quad \csc\theta = \frac{r}{y}$$

$$\textbf{cosine:} \quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:} \quad \sec\theta = \frac{r}{x}$$

$$\textbf{tangent:} \quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:} \quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is acute.

Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if θ is $\pm\pi/2, \pm3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm2\pi, \ldots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.43. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

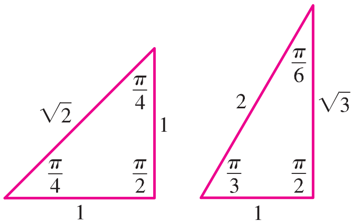


FIGURE 1.43 Radian angles and side lengths of two common triangles.

The CAST rule (Figure 1.44) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.45, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

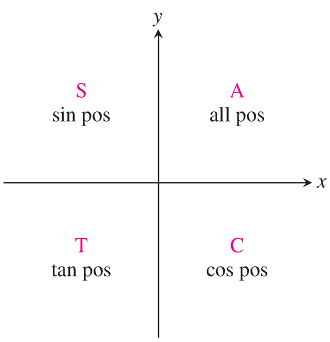


FIGURE 1.44 The CAST rule, remembered by the statement "Calculus Activates Student Thinking," tells which trigonometric functions are positive in each quadrant.

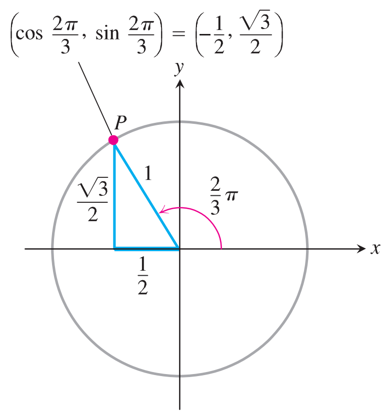


FIGURE 1.45 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Using a similar method we determined the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in Table 1.3.

TABLE 1.3 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + 2\pi) = \tan\theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. Figure 1.46 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π. Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

Even

$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

Odd

$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$

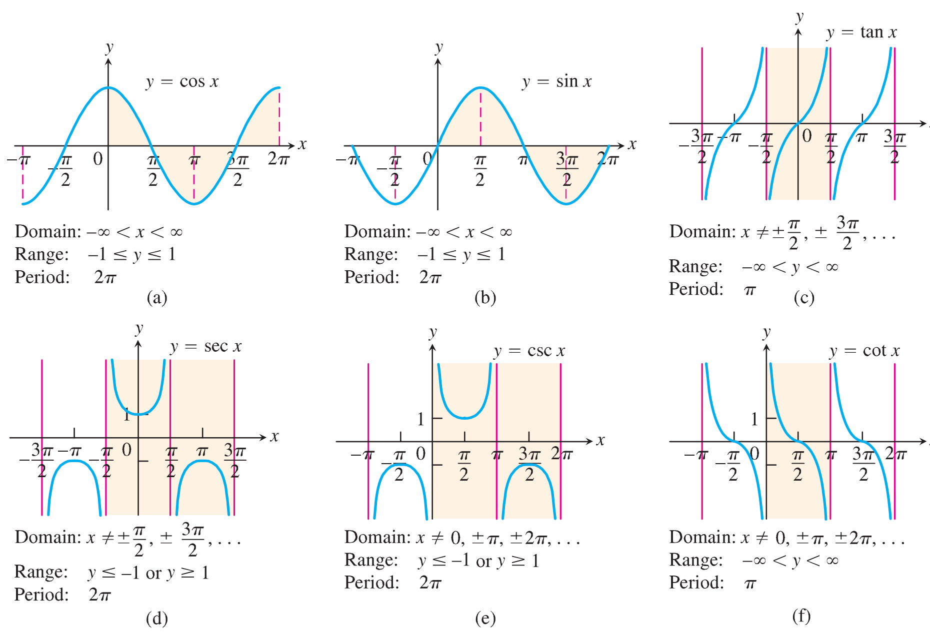


FIGURE 1.46 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x-axis (Figure 1.42). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.47 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \tag{3}$$

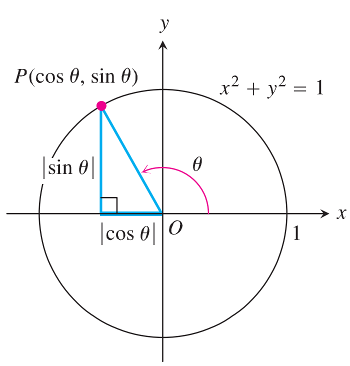


FIGURE 1.47 The reference triangle for a general angle θ.

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The following formulas hold for all angles A and B (Exercise 58).

Addition Formulas

$$\begin{aligned}\cos(A + B) &= \cos A\cos B - \sin A\sin B\\ \sin(A + B) &= \sin A\cos B + \cos A\sin B\end{aligned} \tag{4}$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (3) and (4). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned} \tag{5}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{6}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{7}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{8}$$

This equation is called the **law of cosines**.

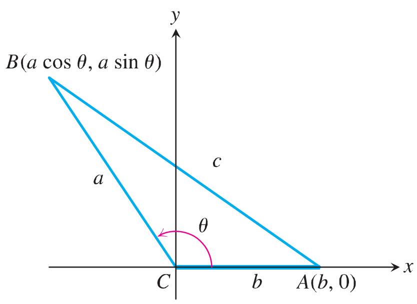


FIGURE 1.48 The square of the distance between A and B gives the law of cosines.

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Figure 1.48. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Transformations of Trigonometric Graphs

The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.

Vertical stretch or compression; reflection about x-axis if negative

Vertical shift

$$y = af(b(x + c)) + d$$

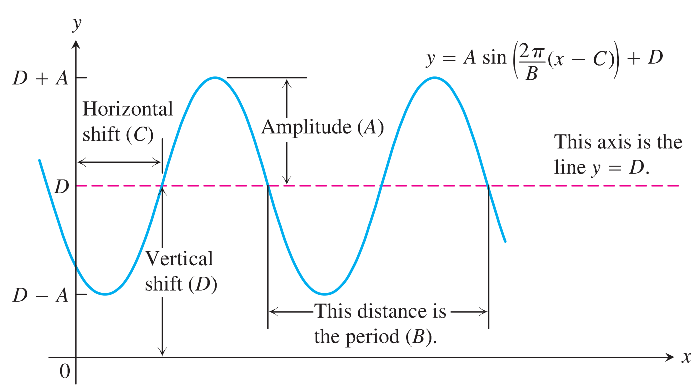
Horizontal stretch or compression; reflection about y-axis if negative

Horizontal shift

The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is revealing and given below.



Two Special Inequalities

For any angle θ measured in radians,

$$\boxed{-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1 - \cos\theta \le |\theta|.}$$

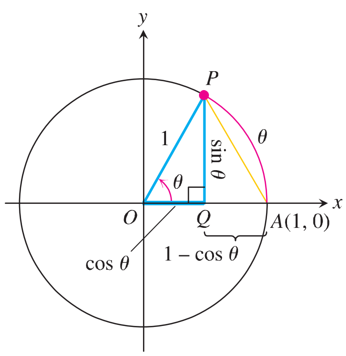


FIGURE 1.49 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 \le \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.49). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2. \tag{9}$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2:

$$\sin^2\theta \le \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 \le \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \qquad \text{and} \qquad |1 - \cos\theta| \le |\theta|,$$

so

$$-|\theta| \le \sin\theta \le |\theta| \qquad \text{and} \qquad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

These inequalities will be useful in the next chapter.

Exercises 1.3

Radians and Degrees

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (**a**) $4\pi/5$ radians? (**b**) $110°$?
2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.
3. You want to make an $80°$ angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \dfrac{3}{5}, \quad x \in \left[\dfrac{\pi}{2}, \pi\right]$
8. $\tan x = 2, \quad x \in \left[0, \dfrac{\pi}{2}\right]$
9. $\cos x = \dfrac{1}{3}, \quad x \in \left[-\dfrac{\pi}{2}, 0\right]$
10. $\cos x = -\dfrac{5}{13}, \quad x \in \left[\dfrac{\pi}{2}, \pi\right]$
11. $\tan x = \dfrac{1}{2}, \quad x \in \left[\pi, \dfrac{3\pi}{2}\right]$
12. $\sin x = -\dfrac{1}{2}, \quad x \in \left[\pi, \dfrac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$
14. $\sin(x/2)$
15. $\cos \pi x$
16. $\cos \dfrac{\pi x}{2}$
17. $-\sin \dfrac{\pi x}{3}$
18. $-\cos 2\pi x$
19. $\cos\left(x - \dfrac{\pi}{2}\right)$
20. $\sin\left(x + \dfrac{\pi}{6}\right)$

21. $\sin\left(x - \frac{\pi}{4}\right) + 1$

22. $\cos\left(x + \frac{2\pi}{3}\right) - 2$

Graph the functions in Exercises 23–26 in the *ts*-plane (*t*-axis horizontal, *s*-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$

24. $s = -\tan \pi t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$

26. $s = \csc\left(\frac{t}{2}\right)$

T **27.** **a.** Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of sec *x* in relation to the signs and values of cos *x*.

b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of csc *x* in relation to the signs and values of sin *x*.

T **28.** Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of cot *x* in relation to the signs and values of tan *x*.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of sin *x* and cos *x*.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$

42. $\cos\left(\frac{3\pi}{2} + x\right)$

43. Evaluate $\sin\frac{7\pi}{12}$ as $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

44. Evaluate $\cos\frac{11\pi}{12}$ as $\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$.

45. Evaluate $\cos\frac{\pi}{12}$.

46. Evaluate $\sin\frac{5\pi}{12}$.

Using the Double-Angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2\frac{\pi}{8}$

48. $\cos^2\frac{5\pi}{12}$

49. $\sin^2\frac{\pi}{12}$

50. $\sin^2\frac{3\pi}{8}$

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ, where $0 \le \theta \le 2\pi$.

51. $\sin^2\theta = \frac{3}{4}$

52. $\sin^2\theta = \cos^2\theta$

53. $\sin 2\theta - \cos\theta = 0$

54. $\cos 2\theta + \cos\theta = 0$

Theory and Examples

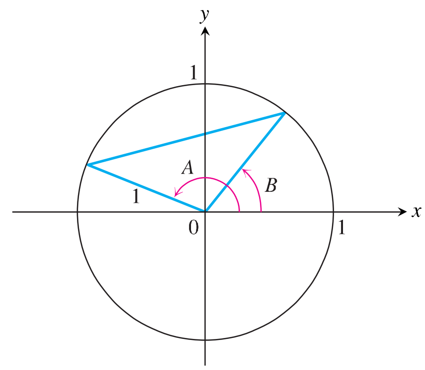
55. The tangent sum formula The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

56. (*Continuation of Exercise 55.*) Derive a formula for $\tan(A - B)$.

57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



58. **a.** Apply the formula for $\cos(A - B)$ to the identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.

b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.

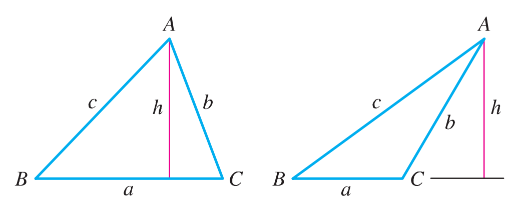
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side *c*.

60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side *c*.

61. The law of sines *The law of sines* says that if *a*, *b*, and *c* are the sides opposite the angles *A*, *B*, and *C* in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.



62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 59). Find the sine of angle B using the law of sines.

63. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

T **64. The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.9, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a. With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b. With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

General Sine Curves

For

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A, B, C, and D for the sine functions in Exercises 65–68 and sketch their graphs.

65. $y = 2\sin(x + \pi) - 1$

66. $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$

67. $y = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$

68. $y = \frac{L}{2\pi}\sin\frac{2\pi t}{L}, \quad L > 0$

COMPUTER EXPLORATIONS

In Exercises 69–72, you will explore graphically the general sine function

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

69. The period B Set the constants $A = 3$, $C = D = 0$.

a. Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b. What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

70. The horizontal shift C Set the constants $A = 3$, $B = 6$, $D = 0$.

a. Plot $f(x)$ for the values $C = 0, 1$, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b. What happens to the graph for negative values of C?

c. What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

71. The vertical shift D Set the constants $A = 3$, $B = 6$, $C = 0$.

a. Plot $f(x)$ for the values $D = 0, 1$, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b. What happens to the graph for negative values of D?

72. The amplitude A Set the constants $B = 6$, $C = D = 0$.

a. Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1, 5$, and 9.

b. What happens to the graph for negative values of A?

1.4 Graphing with Calculators and Computers

A graphing calculator or a computer with graphing software enables us to graph very complicated functions with high precision. Many of these functions could not otherwise be easily graphed. However, care must be taken when using such devices for graphing purposes, and in this section we address some of the issues involved. In Chapter 4 we will see how calculus helps us determine that we are accurately viewing all the important features of a function's graph.

Graphing Windows

When using a graphing calculator or computer as a graphing tool, a portion of the graph is displayed in a rectangular **display** or **viewing window**. Often the default window gives an incomplete or misleading picture of the graph. We use the term *square window* when the units or scales on both axes are the same. This term does not mean that the display window itself is square (usually it is rectangular), but instead it means that the x-unit is the same as the y-unit.

When a graph is displayed in the default window, the x-unit may differ from the y-unit of scaling in order to fit the graph in the window. The viewing window is set by specifying an interval $[a, b]$ for the x-values and an interval $[c, d]$ for the y-values. The machine selects equally spaced x-values in $[a, b]$ and then plots the points $(x, f(x))$. A point is plotted if and